Counterexample for the 2-approximation of finding partitions of rectilinear polygons with minimum stabbing number

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Abstract. This paper presents a counterexample for the approximation algorithm proposed by Durocher and Mehrabi [1] for the general problem of finding a rectangular partition of a rectilinear polygon with minimum stabbing number.

1 Introduction

Given a rectilinear polygon P and a rectangular partition R of P, a segment is said to be **rectilinear** relative to P if it is parallel to one of P's sides. Let s be a maximal rectilinear line segment inside P. The stabbing number of s relative to R is defined as the number of rectangles of R that s intersects. The stabbing number of R is the largest stabbing number of a maximal rectilinear line segment inside P. The Minimum Stabbing Rectangular Partition Problem (**MSRPP**) consists in finding a rectangular partition R of P having the smallest possible stabbing number. Figure 1 illustrates these definitions.

Variants of the problem arise from restricting the set of rectangular partitions that are considered to be valid. One of these variants is called the *conforming case*, in which every edge in the solution must be maximal, i.e., both of its endpoints must touch the border of the polygon. For this problem, in [1], Durocher et al. propose an integer programming model for the conforming case where there are exactly two edges (that can be in the solution) having each reflex vertex as endpoint. Thus, there are also precisely two variables associated to each reflex vertex.

In [1] a 2-approximation algorithm is presented for the conforming case of partitions of rectilinear polygons with minimum stabbing number. That approximation algorithm is based in a rounding of the variables. In the *Conclusion* section of the article, it is stated that the algorithm could be extended for the general case using a formulation described informally and the same rounding rules used in the conforming case.

In this paper we show that the algorithm as described in [1] cannot give a 2-approximation for the general case of the (**MSRPP**). This is done by means of a counterexample to the referred algorithm.

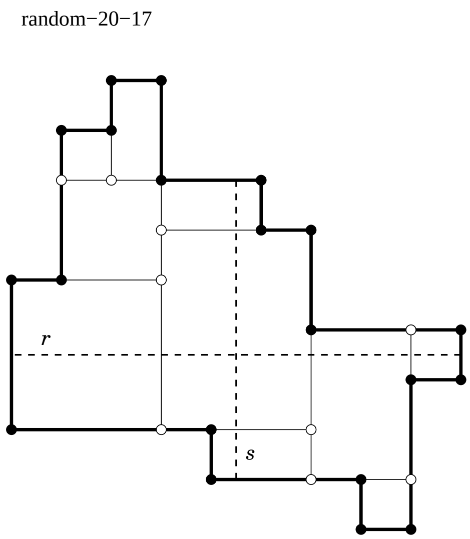

Fig. 1: A rectilinear polygon with a rectangular partition of stabbing number 4. The dashed lines represent maximal rectilinear line segments inside the polygon. Segment r has stabbing number 4 while segment s has stabbing number 3.

2 IP Models

The **MSRPP** can be modelled via integer programming in a number of different ways. In this section we present two such models for the general case of **MSRPP** in an attempt to formalize the description given in [1]. But first, we need some definitions.

Let P be a rectilinear polygon, input of the **MSRPP**. Define as V_r^P the set of **reflex vertices** of P, i.e., those having internal angles equal to $3\pi/2$. Let V_c^P be the set of vertices of P that are not reflex. Denote by $grid(P)$, the set of all maximal rectilinear line segments in the interior of P having a vertex in V_r^P as one of its endpoints. Let V_s^P be the set of points in the intersection of two segments in $grid(P)$. We refer to these points as **Steiner Vertices**. The points that are not in V_r^P or V_c^P and are in the intersection of a segment in $grid(P)$ and the border of P compose the set V_b^P. Denote by V^P the set resulting from the union of all the point sets defined before, i.e., $V^P = V_r^P \cup V_c^P \cup V_s^P \cup V_b^P$.

Define E_h^P as the set of line segments in the border of P having only two points in V^P which are its extremities. Any fragment of a segment in $grid(P)$ containing exactly two vertices in V^P is called an **internal edge**. The set of all internal edges is E_i^P and the set of all edges in P is $E^P = E_h^P \cup E_i^P$. A subset E'^P of E^P defines a *knee* in a vertex $u \in V_s^P \cup V_r^P$ if exactly two edges in E'^P have u as an endpoint and these edges are orthogonal. A subset E'^P of E^P is said to define an *island* in a vertex $u \in V_r^P$ if only one edge of E'^P have u as an endpoint. At last, if ua and ub are two edges in E^P having a common endpoint u, we denote the angle between ua and ub by $\theta(ua, ub)$.

Now, we can formalize the model described in [1] as follows:

$$
\begin{aligned}
(RPST) \qquad z \;=\; & \min k && && (1)\\
\text{subject to} \qquad & x_{ua} + x_{ub} \geq 1, && \forall\, u \in V_r^P \wedge ua, ub \in E_i^P, && (2)\\
& x_{ua} + x_{ub} - x_{uc} \geq 0, && \forall\, u \in V_s^P, \forall\, ua, ub, uc \in E_i^P && \\
& && \text{with } \theta(ua, ub) = \pi/2, && (3)\\
& \sum_{\substack{uv \in E_i^P \\ uv \bigcap s \neq \emptyset}} x_{uv} \leq k - 1, && \forall\, s \in L, && (4)\\
& x_{uv} \in \mathbb{B} && \forall\, uv \in E_i^P, && (5)\\
& k \in \mathbb{Z}. && && (6)
\end{aligned}
$$

In the model above, we have one binary variable x_{uv} for each internal edge uv in P which is set to 1 if and only if the corresponding edge is in the rectangular partition of P. Constraints (2) ensure that the solution does not contain a knee in a reflex vertex. Inequalities (3) impose that the solution does not form a knee or an island in a Steiner vertex. Inequalities (4) relate the x variables with variable k, which represents the stabbing number of the solution. As a consequence, the objective function (1) is to minimize k. Finally, (5) and (6) are integrality restrictions for the variables. Figure 2 shows an instance of the **MSRPP** (called `random-20-17`) with 62 internal edges and their corresponding variables.

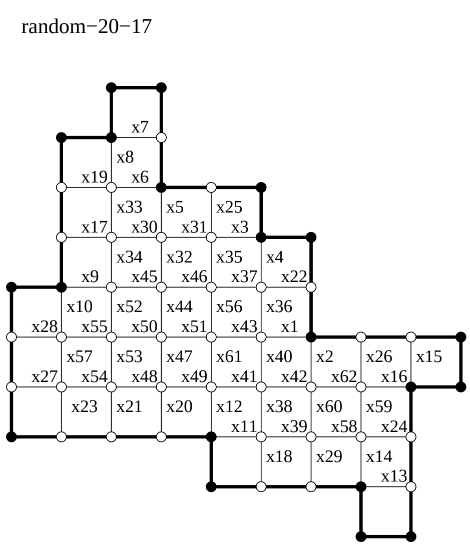

Fig. 2: Instance `random-20-17` with 62 internal edges and the corresponding variables.

As stated before, the ($RPST$) model above is not the only model for the problem and next we show another way of modelling it. However, to guarantee the correctness of the model we must first prove a property of optimal solutions for the **MSRPP**. The following proposition is a generalization of *Observation 1* in [1].

Proposition 1. *Any rectilinear polygon P has an optimal rectangular partition R in which every maximal segment of R has at least one reflex vertex of P as an endpoint.*

Proof. Let R be a rectangular partition of a rectilinear polygon P. Let e be a maximal segment in R having a and b as its endpoints. Suppose neither a nor b are reflex vertices. Since e is maximal and R is a rectangular partition, both endpoints of e must lie in segments perpendicular to e.

Now, since R is a rectangular partition, e define two minimal rectangles (each one possibly containing other rectangles) having e as one of its sides, let us denote them by r_1 and r_2. There are three cases to consider.

The first case consists of r_1 and r_2 been empty rectangles, i.e., neither r_1 nor r_2 contain other rectangles. Therefore, the removal of e unite these rectangles, composing a single rectangle. Therefore, $R \setminus e$ is still a rectangular partition. It is clear that removing a segment cannot increase the stabbing number of the solution. Thus, if R is an optimal solution, so is $R \setminus e$.

The second case to consider is when only one of r_1 or r_2 contains other rectangles. Suppose without loss of generality that r_1 is the one containing other rectangles. Now, we can drag e towards r_1, shrinking any segment with an endpoint in e, until e meets a reflex vertex or the border of P. In the latter case, e is merged to the border of P. It is easy to see that the result of this dragging operation is also a rectangular partition besides, the only stabbing segments affected by this operation are the ones parallel to e and their stabbing number cannot increase. Therefore, as R is optimal, so must be the new solution.

At last, we must consider the case where both r_1 and r_2 contain other rectangles. Suppose without loss of generality that the number of segments in r_1 having an endpoint in e (thus, perpendicular to it) is greater or equal than the number of segments with these characteristics in r_2. Then, again, we can drag e towards r_1, shrinking any segment with an endpoint in e, until e meets either a segment parallel to e or a reflex vertex or the border of P. If a parallel segment is met, e is merged to it and the process is repeated until a reflex vertex or the border of P is met. In case the border of P is met, e ceases to exist together with the segments in the space between e and the border. Once again, the dragging operation results in a rectangular partition of P and the only stabbing segments affected by this operation are parallel to e. But, as the number of segments in r_1 is greater or equal than the number of segments in r_2, one can see that the stabbing number of the new rectangular partition cannot be greater than that of R.

Ergo, there is always an optimal rectangular partition where every maximal segment has at least one reflex vertex of P as an endpoint. $\square$

In the next model, given the same definitions as before, we consider the set E_e^P of rectilinear segments uv where $u \in V_r^P$ and $v \in V^P$. Notice that a segment of E_e^P can be comprised of several consecutive segments of E_i^P. Hence, we call E_e^P the **extended edge** set. In the formulation below, we have a variable x_{uv} for each edge in E_e^P and from Proposition 1 it is easy to notice that this set of variables is sufficient to provide optimal rectangular partitions.

$$
\begin{aligned}
(RPST2) \quad z &= \min k && && (7)\\
\text{subject to} \quad & \sum_{ua \in E_e^P} x_{ua} \geq 1, && \forall\, u \in V_r^P && (8)\\
& x_{ab} + x_{uv} \leq 1, && \forall\, ab, uv : ab \cap uv \neq \emptyset \,\wedge && \\
& && \wedge\, ab \cap uv \neq a, b, u \text{ or } v && (9)\\
& \sum_{\substack{\theta(uv,ab)=\pi/2 \,\wedge \\ \wedge\, b \in uv \,\wedge\, b \neq u \,\wedge\, b \neq v}} x_{uv} - x_{ab} \geq 0, && \forall\, a \in V_r^P, b \in V_s^P && (10)\\
& \sum_{uv \in E_e^P : uv \bigcap s \neq \emptyset} x_{uv} \leq k - 1, && \forall\, s \in L && (11)\\
& x_{uv} \in \mathbb{B} && \forall\, uv \in E_e^P. && (12)\\
& k \in \mathbb{Z} && && (13)
\end{aligned}
$$

In this model, inequalities (8) guarantee that the solution does not contain a knee in a reflex vertex. Constraints (9) enforce planarity (two segments of the partition can only intersect at their extremes). Constraints (10) prevent the existence of knees and islands in a Steiner vertex. Finally, (11) are the stabbing constraints and (12) and (13) are integrality constraints. Figure 3 shows instance `random-20-17` with 42 internal edges and the corresponding variables.

3 The Counterexample

Before discussing the counterexample, we first present the rounding scheme proposed in [1] for the conforming case. Once the optimum of the linear relaxation is computed, the rules for rounding variables in the conforming case are really simple: a variable corresponding to a horizontal segment is rounded down to zero if its value is smaller than or equal to 0.5 and is rounded up to one if its value is greater than 0.5. A variable corresponding to a vertical segment is rounded down to zero if its value is smaller than 0.5 and is rounded up to one if its value is greater than or equal to 0.5.

In the *Conclusion* section of [1], a model for the general (non-conforming) case is described informally. From the discussion, apparently such model is equivalent to the $(RPST)$ formulation given in Section 2. According to the authors, the same rounding rules used in the conforming case provide a 2-approximation for the general case.

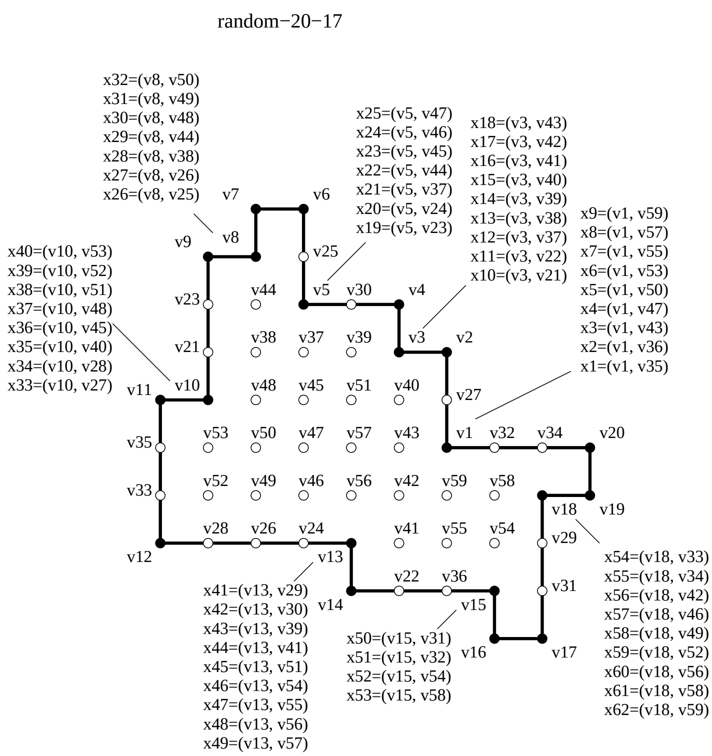

Fig. 3: Instance `random-20-17` with its extended edges and corresponding variables.

The rounding rules do not mention what should be done for Steiner vertices, and no guarantee is given that applying them directly in these situations will avoid the formation of a knee or an island. In fact, the instance displayed in Figure 4 shows that this cannot always be done without sacrificing feasibility. In this figure, the optimal values of the variables corresponding to edges incident to Steiner vertex `v39` (see Figure 3) after solving the linear relaxation associated to instance `random-20-17` are given. As only the variable corresponding to one vertical edge incident to that vertex has value equal to 0.5 and the other three are smaller than 0.5, rounding according to that rule would result in an island at $v37$. Therefore, the set of edges obtaining after rounding does not form a rectangular partition.

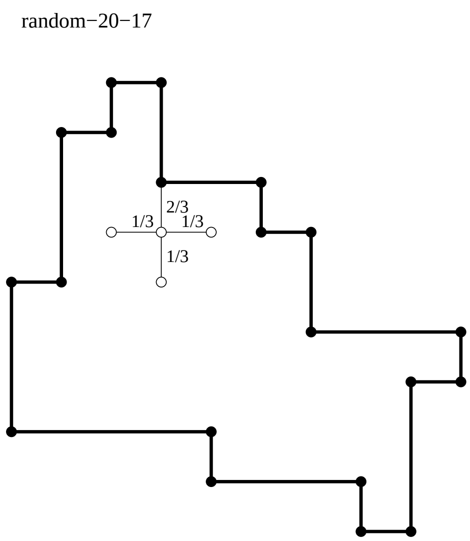

Fig. 4: Values of variables corresponding to edges incident to a Steiner vertex after solving linear relaxation.

It is however possible that we misinterpreted the model the authors were thinking of (although there is evidence in contrary) and the idea is actually to define variables corresponding to all edges having a reflex vertex as one of its endpoints. If so, the formulation would look like $(RPST2)$ model in the previous section. In this alternative formulation, rounding the variables using that rule does not cause the same problem as before since every variable correspond to an edge having a reflex vertex as endpoint.

Contrary to what happens in the conforming case, however, the reflex vertices here have more than two incident edges. Therefore, it is possible that the solution of the linear relaxation result in values smaller than 0.5 for all the variables corresponding to the edges incident to a certain reflex vertex. Thus, the rounding of such solution would result in a partition having a knee in a reflex vertex.

The situation described above occurs in practice with instance `random-20-17`, as shown in Figure 5. Consider the edges incident to vertex `v5`. All the associated variables incident to this vertex have value smaller than 0.5. As consequence, they will be rounded to zero, resulting in the formation of a knee at `v5` and, therefore, in an infeasible solution.

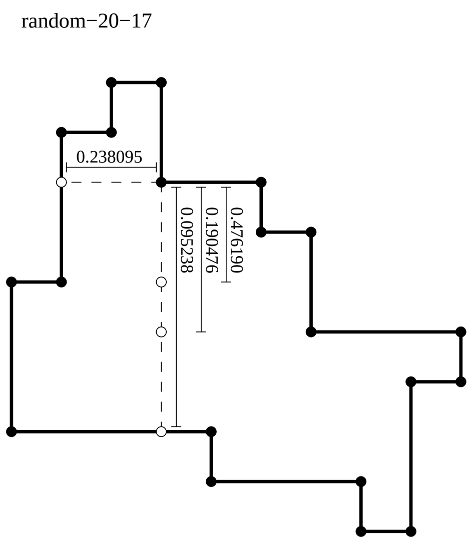

Fig. 5: Values of variables corresponding to edges incident to a reflex vertex after solving linear relaxation.

4 Conclusion

From the counterexample presented in Section 3, we conclude that it remains open whether a 2-approximation for the **MSRPP** in the general case exists. It is, however, noteworthy that many other contributions are presented in [1] and none of them are diminished by this counterexample.

References

1. Stephane Durocher and Saeed Mehrabi. Computing partitions of rectilinear polygons with minimum stabbing number. In Joachim Gudmundsson, Julián Mestre, and Taso Viglas, editors, *Computing and Combinatorics*, volume 7434 of *Lecture Notes in Computer Science*, pages 228–239. Springer Berlin Heidelberg, 2012.

Appendix

```
File name: random-20-17.rect
Model: RPST
Vertex number: 59
Edge number: 62

Reading Problem stab
Problem Statistics
         231 (      0 spare) rows
          63 (      0 spare) structural columns
         752 (      0 spare) non-zero elements
Global Statistics
          63 entities        0 sets        0 set members
Minimizing MILP stab
Original problem has:
       231 rows           63 cols          752 elements        63 globals

   Its         Obj Value      S   Ninf  Nneg        Sum Inf  Time
     0           .000000      D      1     0      24.000000     0
    74          2.416667      D      0     0        .000000     0
Optimal solution found
 *** Search unfinished ***    Time: 0
Number of integer feasible solutions found is 0
Best bound is     2.416667

Solution:

X1  =  0.583333  X2  =  0.416667  X3  =  0.095238  X4  =  0.916667  X5  =  0.666667
X6  =  0.333333  X7  =  0.428571  X8  =  0.571429  X9  =  0.904762  X10 =  0.845238
X11 =  1.000000  X12 =  0.571429  X13 =  0.000000  X14 =  1.000000  X15 =  0.000000
X16 =  1.000000  X17 = -0.000000  X18 =  0.500000  X19 =  0.238095  X20 = -0.000000
X21 = -0.000000  X22 =  0.333333  X23 =  0.428571  X24 =  1.000000  X25 =  0.238095
X26 = -0.000000  X27 =  0.285714  X28 =  0.130952  X29 =  0.500000  X30 =  0.333333
X31 =  0.333333  X32 =  0.333333  X33 =  0.333333  X34 =  0.333333  X35 =  0.095238
X36 =  0.583333  X37 =  0.333333  X38 =  0.500000  X39 =  0.500000  X40 =  0.297619
X41 =  0.285714  X42 =  0.583333  X43 =  0.285714  X44 = -0.000000  X45 =  0.333333
X46 =  0.333333  X47 = -0.000000  X48 =  0.285714  X49 =  0.285714  X50 =  0.285714
X51 =  0.285714  X52 =  0.571429  X53 =  0.285714  X54 =  0.285714  X55 =  0.571429
X56 =  0.000000  X57 =  0.714286  X58 =  1.000000  X59 = -0.000000  X60 =  0.500000
X61 =  0.285714  X62 =  1.000000  X63 =  3.000000

*********************************************************************************

File name: random-20-17.rect
Model: RPST2
```

```
Vertex number: 59
Edge number: 62
Reading Problem stab
Problem Statistics
         336 (      0 spare) rows
          63 (      0 spare) structural columns
         996 (      0 spare) non-zero elements
Global Statistics
          63 entities        0 sets        0 set members
Minimizing MILP stab
Original problem has:
       336 rows           63 cols          996 elements        63 globals
Crash basis containing 13 structural columns created

   Its         Obj Value      S   Ninf  Nneg        Sum Inf  Time
     0           .000000      D      1     0      24.000000     0
    66          2.413793      D      0     0        .000000     0
Optimal solution found
 *** Search unfinished ***    Time: 0
Number of integer feasible solutions found is 0
Best bound is     2.413793

Solution:

X1  =  0.190476  X2  =  0.380952  X3  =  0.142857  X4  = -0.000000  X5  = -0.000000
X6  = -0.000000  X7  = -0.000000  X8  = -0.000000  X9  =  0.285714  X10 = -0.000000
X11 =  0.142857  X12 =  0.142857  X13 = -0.000000  X14 =  0.000000  X15 =  0.523810
X16 = -0.000000  X17 = -0.000000  X18 =  0.190476  X19 =  0.238095  X20 =  0.095238
X21 =  0.000000  X22 = -0.000000  X23 =  0.476190  X24 = -0.000000  X25 =  0.190476
X26 =  0.380952  X27 =  0.190476  X28 =  0.000000  X29 =  0.238095  X30 = -0.000000
X31 = -0.000000  X32 =  0.190476  X33 =  0.523810  X34 =  0.809524  X35 = -0.000000
X36 =  0.285714  X37 =  0.380952  X38 =  0.000000  X39 =  0.000000  X40 =  0.190476
X41 =  0.619048  X42 = -0.000000  X43 = -0.000000  X44 =  0.142857  X45 = -0.000000
X46 = -0.000000  X47 = -0.000000  X48 =  0.380952  X49 =  0.000000  X50 =  0.380952
X51 = -0.000000  X52 =  0.619048  X53 = -0.000000  X54 = -0.000000  X55 =  0.095238
X56 =  0.142857  X57 = -0.000000  X58 = -0.000000  X59 =  0.380952  X60 = -0.000000
X61 =  0.000000  X62 =  0.380952  X63 =  3.000000
```